Velociraptor Pursuit Problem

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Abstract

The principle object of this paper is to demonstrate how to solve a specific pursuit curve problem using differential equations. Pursuit curves involve a pursuer and a pursuee where the curve of pursuit is the curve traced by the pursuer. The pursuer must always be heading in the direction of the thing it is chasing’s position and does not anticipate where it will be. Methods of finding this curve of pursuit (position function of the pursuer) will be investigated in this paper.
Contents

1 Pursuit Curves ................................................. 4

2 Description of the Problem .................................. 4

3 Explicit Solution Without Acceleration ....................... 6
   3.1 Setting up the Problem .................................... 6
   3.2 Finding the Differential Equation ......................... 9
   3.3 Solving the Differential Equation using Mathematica ..... 11
   3.4 Conclusion .................................................. 13

4 Numerical Solution with Acceleration ....................... 13
   4.1 Setting Up the Problem .................................... 13
   4.2 Finding the Differential Equations ....................... 13
   4.3 Solving the Differential Equations Using Mathematica ..... 15
   4.4 Conclusion .................................................. 24
1 Pursuit Curves

The pursuit curve is a mathematical model that describes a specific relationship of two or more objects. It has often been found that the curve of pursuit was once the “work of Leonardo da Vinci, and then moved to a Frenchman, Pierre Bouguer (1698-1758) who expanded pursuit to two dimensions” (Pursuit). In journal entries by J.C. Barton, C.J. Eliezer, and C.C. Puckette, they discuss the curve of pursuit that was evidently initially posed by Leonardo da Vinci around 1510. It must also be understood that there have been problems dealing with the various aspects of the curve of pursuit, including falconry in Europe and Asia as far back as 2000 BC; however, the first mathematical solution to be accepted by the educated world was that of Pierre Bouguer in 1732.

In the early 1700s Pierre Bouguer derived the solution of the path of a ship pursuing another at a constant speed, always changing course to follow the lead ship. Publishing his findings in 1732, “[calling] the path of the pursuing ship the Courbe (or Ligne) de Poursuite, and that of the one pursued the Ligne de Fuite” (Puckette), Bouguers investigation of the curve is often considered a result of his main work in the field of navigation and hydrography. Sometime after Pierre published his solution, P.L.M. de Maupertuis analyzed the problem and also solved it; however, it is stated that, “His solution of this is left as a differential equation, into which are to be substituted the given functions did not work well, for it complicated even the straight line case” (Puckette). Although de Maupertuis’ answer was not practical, it shed light on the different types of curves and how they behaved.

The first definition of the term ‘pursuit curve’ was not until 1859 in George Booles “Treatise on differential equations”. “the term curve of pursuit is given to the path which a point describes when moving with uniform velocity towards another point which moves with uniform velocity in a given curve” (Barton). Booles solution yields a two dimensional second-order differential equation for “the curved path described by a fighter plane making an attack on a moving target while holding the proper aiming allowance” (Lloyd). With Bouguers and Booles solutions to problems dealing with two dimensional curves of pursuit, a general solution to three dimensional pursuit curve problems could then easily be derived and implemented for any case of pursuit.

2 Description of the Problem

The problem that we will attempt to solve is the second question of this comic.
1. The velociraptor spots you 40 meters away and attacks, accelerating at 4 m/s² up to its top speed of 25 m/s. When it spots you, you begin to flee, quickly reaching your top speed of 6 m/s. How far can you get before you’re caught and devoured?

2. You are at the center of a 20m equilateral triangle with a raptor at each corner. The top raptor has a wounded leg and is limited to a top speed of 10 m/s.

(Not to scale)

The raptors will run toward you. At what angle should you run to maximize the time you stay alive?

3. Raptors can open doors, but they are slowed by them. Using the floor plan on the next page, plot a route through the building, assuming raptors take 5 minutes to open the first door and halve the time for each subsequent door. Remember, raptors run at 10 m/s and they do not know fear.
From the comic we can now create a list of parameters for the problem. First, we know that the person is standing in the center a 20 meter equilateral triangle formed by the raptors. By doing some simple geometry we come to the understanding that each raptor is $\frac{20}{\sqrt{3}}$ meters away from the person. We also know that the person travels at 6 m/s instantaneously from the start with no acceleration. On the other hand, each raptor runs with an acceleration of 4 m/s$^2$ and starts with an initial velocity of 0 m/s. To make things interesting they have also decided that the top raptor is injured and therefore maxes out at 10 m/s whereas the other two raptors on the bottom max out at 25 m/s.

The last thing that the problem specifies is that the velociraptors run toward the person, meaning that they are always pointed in the person’s direction. The raptors will never anticipate where they are going. This means that the position of the each raptor completely relies on the path the person chooses to take. For the problem, the person will pick a straight path to run that will be measured by an angle from the positive x-axis. The solution to this problem will be the angle that maximizes the person’s time of survival before one of the raptors catches and eats them.

Something else we will use to our advantage throughout the solutions to this problem is that this problem is symmetrical about the y-axis. This means that we can disregard one of the raptors on the bottom and only worry about one side of the problem. In this report, we will focus our efforts on the angle between the top raptor, whom we may refer to as raptor 1 sometimes, and the raptor on the bottom right, whom we will refer to as raptor 2.

3 Explicit Solution Without Acceleration

3.1 Setting up the Problem

Solving a problem explicitly is a very convenient way of solving a problem due to the fact that it will give an equation that gives an exact answer to the problem. Unfortunately in this case, in order to solve the problem explicitly, we will have to ignore the acceleration. The reason why will be seen when setting up the differential equations. Since there will be no acceleration used in the process of finding the solution we will assume that the raptors both start off at their maximum velocities of 10 m/s and 25 m/s.

Also, in order to make this problem easier to solve, we must rotate the graph so that the runner is always running straight up. We will then shift it over one so that the runner’s path will be along $x = 1$ for the entire duration of the run. Since we have now rotated and shifted the graph the raptors will have new initial positions on our new cartesian plane. Here is the graph just before it is shifted.
As you can see, the red lines represent the original position of the graph and the dotted green lines are where it will be rotated and shifted to. The blue line represents the path of the person running away from the raptors. You can see that the blue line will be set at an angle of theta above the positive x-axis. When the graph is rotated and shifted so that the person is running straight up $x = 1$ it will look something like this.
The green lines now show where the graph will be after being rotated and shifted and the dotted red lines show where the graph used to be. As you can see, the angle between the negative x-axis and the line going from the person to raptor 1 is the same angle that the person chooses to run at, which we found using some geometry techniques. Another angle that was found through geometry is the one between the old positive x-axis and the line connecting the person to raptor 2. This angle is 30 degrees. Therefore, the angle between the new positive y-axis and the line connecting the person to raptor 2 is $\theta + 30^\circ$ or in radians $\theta + \frac{\pi}{3}$. Knowing these angles will now help us find the initial positions of the raptors using the light blue right triangles you see in the picture above. For example, if $\theta$ were equal to 30 degrees they would look something like this.
Using some trigonometry we can then conclude that the initial position of raptor 1 (the top raptor) is

\[
\left(1 - \cos(\theta) \left(\frac{20}{\sqrt{3}}\right), \sin(\theta) \left(\frac{20}{\sqrt{3}}\right)\right)
\]

and the initial position of raptor 2 (the bottom raptor) is

\[
\left(1 + \sin\left(\frac{\pi}{3} + \theta\right) \left(\frac{20}{\sqrt{3}}\right), \cos\left(\frac{\pi}{3} + \theta\right) \left(\frac{20}{\sqrt{3}}\right)\right)
\]

These initial positions are important information that we will use as initial conditions when solving our differential equations.

### 3.2 Finding the Differential Equation

The next thing we must do to solve this problem is come up with our differential equation. The one thing we know to start off with is that the velociraptor is always going to be heading towards the person no matter where the person is. This means that the slope from the raptors position to the position of the person at any given point in time must be equal to the instantaneous rate of change of the position of the raptor which would be the direction its moving towards. This makes sense considering that the raptor is always directed towards the person and since the person’s position is always changing it will directly correlate with the position of the raptor. We can express this idea with this equation.

\[
\frac{dy}{dx} = \frac{y - 6t}{x - 1}
\]

As you can see this is a basic slope formula but instead of numbers we have the person’s position as a function of time. We say that the person is traveling vertically
up the line \( x = 1 \) due to the changed graph and will take into account the angle at which they run with the initial conditions of the raptors.

Now we must solve for \( t \) so that we can get rid of it as you will see in a later step.

\[
t = \frac{y - (x - 1)(\frac{dy}{dx})}{6}
\]

We will then use the arc length formula as a way to get rid of \( t \). The arc length formula in this instance is basically telling us that regardless of what path the raptor takes he will still travel a certain distance along his arc and that distance will be equal to his speed multiplied by the time that has passed. This step right here is why we cannot use acceleration to solve this particular differential equation. If we were to introduce the acceleration of the raptors right now it would cause an equation too complex for even mathematica to solve. So instead, we continue using the maximum velocity of 10 m/s for the top raptor.

\[
10t = \int_{0}^{x} \sqrt{1 + \left[y'(u)\right]^2} du
\]

Once we have solved for both \( t \)'s in the two equations we can set them equal to each other and now we have an equation entirely in terms of \( y \) and \( x \) which if you haven’t been able to figure out yet are the \( x \) and \( y \) components of the raptors position.

\[
\frac{y - (x - 1)(\frac{dy}{dx})}{6} = \frac{\int_{0}^{x} \sqrt{1 + \left[y'(u)\right]^2} du}{10}
\]

Next, in order to get rid of the integral on the right hand side, we will take the derivative of both sides with respect to \( x \).

\[
-\frac{(x - 1)(\frac{dy}{dx})}{6} = \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{10}
\]

With this equation alone we will not be able to solve the differential equation. To help us out we must let \( w = \frac{dy}{dx} \)

\[
(x - 1)\frac{d}{dx} w = \left(\frac{6}{10}\right)\sqrt{1 + w^2}
\]

After that change you can see that we now have a differential equation that we can solve. To do so, we will use mathematica’s command DSolve.
3.3 Solving the Differential Equation using Mathematica

First we will solve the differential equation from the previous page. In order to do so in mathematica we will need to address some initial conditions for our first raptor. We will do this using its initial position. As discussed in the introduction, the position of the raptor on top when the axis is rotated is

\[ \left( 1 - \cos(\theta) \left( \frac{20}{\sqrt{3}} \right), \sin(\theta) \left( \frac{20}{\sqrt{3}} \right) \right) \]

We will also use the fact that the slope of the raptor’s initial trajectory is

\[ -\left( \frac{\sin(\theta)}{\cos(\theta)} \right) = -\tan(\theta) \]

We must use the slope of the raptor’s initial trajectory as our first initial condition since our first differential equation is in terms of \( w(x) \) which is the slope of the raptor. Plugging that into mathematica looks like this:

\[
\text{DSolve}\left[ \left\{ (x-1) w'[x] = -\frac{6}{10} \sqrt{1 + w[x]^2}, w\left[ 1 - \cos[a] \cdot \frac{20}{\sqrt{3}} \right] = -\tan[a] \right\}, w[x], x \right]
\]

That then gives us an equation:

\[
\left\{ \sinh\left[ \frac{1}{5} \left( -5 \text{ArcSinh}[\tan[a]] - 3 \log[5 - 5 x] + 3 \log[5 - 5 \left( 1 - \frac{20 \cos[a]}{\sqrt{3}} \right) ] \right) \right] \left[ w[x] \rightarrow \text{Sinh}\left[ \frac{1}{5} \left( -5 \text{ArcSinh}[\tan[a]] - 3 \log[5 - 5 x] + 3 \log[5 - 5 \left( 1 - \frac{20 \cos[a]}{\sqrt{3}} \right) ] \right) \right] \right] \right\}
\]

As you may recall, \( w = \frac{dy}{dx} \) so now we are able to solve the differential equation for \( y(x) \). Once again we will use the initial conditions as the starting position of the raptor,

\[ \left( 1 - \cos(\theta) \left( \frac{20}{\sqrt{3}} \right), \sin(\theta) \left( \frac{20}{\sqrt{3}} \right) \right) \]

\[
\text{DSolve}\left[ \left\{ y'[x] = \sinh\left[ \frac{1}{5} \left( -5 \text{ArcSinh}[\tan[a]] - 3 \log[5 - 5 x] + 3 \log[5 - 5 \left( 1 - \frac{20 \cos[a]}{\sqrt{3}} \right) ] \right) \right], \right\}
\]

The equation that this gives us is now in terms of \( y(x) \) or the position of the raptor. Since the person is traveling at a speed of 6 m/s and directly up \( x = 1 \), it can be
said that \( y(x) = 6t \) and \( x = 1 \) for the position of the person. We can then substitute these values into the \( y(x) \) and \( x \) of the raptor’s position in order to find out where they intersect. Then you can solve for \( t \) and see at what time the raptor catches you based on which angle you chose to take. This is due to the fact that our answer was in terms of \( \theta \) since we used it in the raptor’s initial conditions. So once these values are substituted in we get a function in terms of \( t(\theta) \). Here is an example of the graph with \( \theta \) (radians) being on the x-axis and \( t \) (seconds) being on the y-axis.

We then do the same for the Second raptor who’s initial position is

\[
\left( 1 + \sin \left( \frac{\pi}{3} + \theta \right) \left( \frac{20}{\sqrt{3}} \right), \cos \left( \frac{\pi}{3} + \theta \right) \left( \frac{20}{\sqrt{3}} \right) \right)
\]

and initial trajectory is

\[
\frac{\cos(\theta)}{\sin(\theta)} = \cot(\theta)
\]

Also recall that this raptor’s speed is 25m/s which is taken into account when generating the differential equation. Once all this is done the plot of the two functions on one another looks like this:
We choose to look only at the angles that lie between the two raptors since any other angle would be out of the question for this problem. It is then understood that the point where these two graphs intersect is where the time of survival is maximized. This is due to the fact that at any other angle on this graph one raptor would catch you before the other one has the chance and therefore when they catch you at the same time the time is the greatest. You can see this visually on the graph. This point of intersection calculated in mathematica was about \((.57, 1.23)\) or in other terms at \(33^\circ\) in 1.23 seconds.

### 3.4 Conclusion

From our results we can conclude that disregarding acceleration and having the raptors start off at their maximum velocities, the ideal angle to run from the positive x-axis would be approximately \(33^\circ\) based on the results of our explicit solution in mathematica. At this angle you would last for only 1.23 seconds. From this exercise we learned how to calculate an explicit solution using differential equations.

### 4 Numerical Solution with Acceleration

#### 4.1 Setting Up the Problem

To solve the problem numerically we would not need to rotate or shift the graph this time. So first we start off by finding the initial positions of the raptors. This was not nearly as hard this time and we found the initial position of the top raptor to be

\[
(0, \frac{20}{\sqrt{3}})
\]

and the initial position of the bottom raptor to be

\[
(10, -\frac{10}{\sqrt{3}})
\]

given that the angle separating the bottom raptor and the person is \(30^\circ\) below the positive x-axis.

These were really the only things we needed to set up the problem and so we move on to finding the differential equations.

#### 4.2 Finding the Differential Equations

Solving this problem numerically we used a completely different approach then what we did to solve it explicitly. Here, instead of finding a differential equation by getting rid of the time component, we use time to our advantage and set up a series
of parametric differential equations. Finding these equations was not easy though. First we found the parametric equations of the human. This was rather easy since the human has a distance of \( d = 6t \) from the origin given that they travel at 6 m/s. We also know that they will be running at an angle of \( \theta \) above the positive x-axis. Therefore, using simple trigonometry, we conclude that the parametric equations for the human’s x and y position are

\[
x_p(t) = 6t\cos(\theta)
\]
\[
y_p(t) = 6t\sin(\theta)
\]

Together we can say that the humans vector is \( \mathbf{P}(t) \). Next we use these to find the position vector of the raptors. Let’s say that the raptor’s position is in terms of \( x_r(t) \) and \( y_r(t) \). We will call this vector \( \mathbf{R}(t) \). Once again, we know that the direction of the velocity of the raptor will be located directly towards the position of the human by the definition of pursuit curves. Therefore we do the same thing as before and say

\[
\mathbf{R}'(t) = \mathbf{P}(t) - \mathbf{R}(t)
\]

We also know that the speed of the raptor is \( 4t \) since its accelerating at \( 4 \text{ m/s}^2 \).

Using our knowledge of vectors and physics we can then say that this equals the magnitude of the vector \( \mathbf{R}'(t) = \mathbf{P}(t) - \mathbf{R}(t) \). Since they’re both equal we can say that

\[
1 = \frac{4t}{|\mathbf{P}(t) - \mathbf{R}(t)|}
\]

Multiplying this into the right hand of the equation of above we get

\[
\mathbf{R}'(t) = \frac{4t(\mathbf{P}(t) - \mathbf{R}(t))}{|\mathbf{P}(t) - \mathbf{R}(t)|}
\]

Using the pythagorean theorem and our definition of a vector’s magnitude we then get that

\[
|\mathbf{P}(t) - \mathbf{R}(t)| = \sqrt{(x_p(t) - x_r(t))^2 + (y_p(t) - y_r(t))^2}
\]

Now we can expand our equation and get two parametric differential equations.

\[
x_r'(t) = \frac{4t \left( x_p(t) - x_r(t) \right)}{\sqrt{(x_p(t) - y_r(t))^2 + (x_p(t) - x_r(t))^2}}
\]
\[
y_r'(t) = \frac{4t \left( y_p(t) - y_r(t) \right)}{\sqrt{(x_p(t) - y_r(t))^2 + (x_p(t) - x_r(t))^2}}
\]

We can then plug in the known values of the human’s position and this gives us:

\[
x_r'(t) = \frac{4t \left( 6t \cos(\theta) - x_r(t) \right)}{\sqrt{(6t \cos(\theta) - x_r(t))^2 + (6t \sin(\theta) - y_r(t))^2}}
\]
\[ y'(t) = \frac{4t \left( 6t \sin(\theta) - y_r(t) \right)}{\sqrt{(6t \cos(\theta) - x_r(t))^2 + (6t \sin(\theta) - y_r(t))^2}} \]

We now have our two differential equations. Next we needed to make sure to add our initial conditions which were the initial position of the raptor.

\[ x(0) = 0 \]
\[ y(0) = \frac{20}{\sqrt{3}} \]

The next thing we needed to do was to take into account that the top raptor had a max velocity of 10 m/s. To do so we calculated at what time the raptor would reach this velocity. We found this to be at \( t = 2.5 \). So within the mathematica code we knew we needed to make a piecewise function with the velocity being \( 4t \) when \( t < 2.5 \) and 10 when \( t \geq 2.5 \).

All of this was the same for the other raptor as well, the only things that changed were the initial conditions and the velocity at which it maxed out at. Since his maximum speed was 25 m/s we calculated that it would reach that at \( t = 6.25 \). Therefore in its piecewise function we would make sure the speed was \( 4t \) when \( t < 6.25 \) and 25 when \( t \geq 6.25 \). Also if you recall the bottom raptor’s starting position, the initial conditions would be,

\[ x(0) = 10 \]
\[ y(0) = -\left( \frac{10}{\sqrt{3}} \right) \]

### 4.3 Solving the Differential Equations Using Mathematica

This time since we are solving numerically we will use the mathematica code NDSolve. This command allows you to find answers to your differential equation along a range of values for one of the variables which in this case is \( t \). You can then plot this in terms of \( y(x) \) on a plot using what they call an Interpolating Function. For this though we cannot let \( \theta \) go unattended to. Therefore we must plug in a different value of \( \theta \) every time we run the command. This is what the command looked like for the top and bottom raptors:
From here we plugged in angles ranging from -20° to 80° increasing by 10° each time in the mathematica code where θ is represented as a. Here are the graphs of the positions of the raptors and the position of the person up until one of the raptors catches the person.

\[
\begin{align*}
X'[t] &= \text{Piecewise}\left[\left\{ \frac{4 \cdot (6 \cdot \cos(a) - x[t])}{\sqrt{(6 \cdot \cos(a) - x[t])^2 + (6 \cdot \sin(a) - y[t])^2}}, \quad t < 6.25 \right\}, \left\{ \frac{25 \cdot (6 \cdot \cos(a) - x[t])}{\sqrt{(6 \cdot \cos(a) - x[t])^2 + (6 \cdot \sin(a) - y[t])^2}}}, \quad t > 6.25 \right\}\right], \\
Y'[t] &= \text{Piecewise}\left[\left\{ \frac{4 \cdot (6 \cdot \sin(a) - y[t])}{\sqrt{(6 \cdot \cos(a) - x[t])^2 + (6 \cdot \sin(a) - y[t])^2}}, \quad t < 6.25 \right\}, \left\{ \frac{25 \cdot (6 \cdot \sin(a) - y[t])}{\sqrt{(6 \cdot \cos(a) - x[t])^2 + (6 \cdot \sin(a) - y[t])^2}}}, \quad t > 6.25 \right\}\right], \\
x[0] &= 10, \quad y[0] = -\frac{10}{\sqrt{3}}, \quad \{a[t], y[t]\}, \{t, 5\}
\end{align*}
\]
$10^\circ$

$20^\circ$

$30^\circ$
We then took all the times that it took each raptor to catch the person and plotted the data points on a graph.

As you can see this is the same kind of graph we dealt with earlier. Our goal now was to find the intersection of these two plots since that would maximize the time of
escape. Since it looked like they intersected between about 30° and 40° we decided to plot some more points between this interval. The graph came out like this.

With this many data points we felt ready to find an equation using mathematica that would best represent this data. It looked like this.

We then set the two best fit equations equal to one another and got an answer of (32.79,3.1) or 3.1 seconds at 32.79°.
4.4 Conclusion

From our results we can conclude that including acceleration and obeying all parameters of the problem, the ideal angle to run from the positive x-axis would be approximately 32.79° based on the results of our numerical solution in mathematica. At this angle you would last for about 3.1 seconds. From this exercise we learned how to calculate a numerical solution using differential equations and mathematica.
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